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Extension of hereditary symmetry operators

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Abstract. Two models of candidates for hereditary symmetry operators are proposed and thus many nonlinear systems of evolution equations possessing infinitely many commuting symmetries may be generated. Some concrete structures of hereditary symmetry operators are carefully analysed on the basis of the resulting general conditions and several corresponding nonlinear systems are explicitly given as illustrative examples.

1. Introduction

An application of Lax pairs is a well known way to construct nonlinear integrable systems. Most integrable systems, such as the Korteweg–de Vries (KdV), the nonliner Schrödinger (NLS), the Kadomtsev–Petviashvili (KP) and the Davey–Stewartson equations, can be derived through appropriate Lax pairs (see, for example, [1]). There are also some other ways to construct nonlinear integrable systems, for example by bi-Hamiltonian formulation [2, 3] and by hereditary symmetry operators [4, 5] etc.

Of course, integrable systems generated by different methods have different integrable properties. In general, the method of Lax pairs produces S-integrable systems and the methods of bi-Hamiltonian formulation and hereditary symmetry operators produce nonlinear systems possessing infinitely many symmetries and/or infinitely many conserved densities. There has already been a lot of investigation on the method of Lax pairs (see, for example, [6]) and the method of bi-Hamiltonian formulation (see, for example, [7–9]). So far, however, there has been little discussion about the method of hereditary symmetry operators.

This paper will focus on the construction of hereditary symmetry operators and their related nonlinear systems. The resulting nonlinear systems have infinitely many commuting symmetries. Some such systems may be found in [10–13]. However, we can easily construct as many such systems as we want. To achieve our aim we first discuss the structure of hereditary symmetry operators by examining two models of candidates for hereditary symmetry operators, and then exhibit some concrete examples of hereditary symmetry operators including relevant nonlinear systems.

Let u be a dependent variable $u = (u^1, \ldots, u^q)^{\mathrm{T}}$, where u^i, $1 \leqslant i \leqslant q$, depend on the spatial variable x and on the temporal variable t. We use $\mathcal{A}^q$ to denote the space of q-dimensional column vector functions depending on u itself and its derivatives with respect to the spatial variable x (possibly a vector). Sometimes we write this space as $\mathcal{A}^q(u)$ in order to show the dependent variable u.

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Definition 1.1. Let $K, S \in \mathcal{A}^q$ and $\Phi(u) : \mathcal{A}^q \to \mathcal{A}^q$. Then the Gateaux derivatives of K and Φ with respect to u at the direction S are defined as

$$K'(u)[S] = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} K(u+\varepsilon S) \qquad \Phi'(u)[S] = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \Phi(u+\varepsilon S). \tag{1.1}$$

We recall that the commutator between two vector functions $K, S \in \mathcal{A}^q$ is given as

$$[K, S] = K'(u)[S] - S'(u)[K]. \tag{1.2}$$

The space $\mathcal{A}^q$ constitutes a Lie algebra under the bilinear operation (1.2).

Definition 1.2. A linear operator $\Phi(u) : \mathcal{A}^q \to \mathcal{A}^q$ is called a hereditary symmetry operator [14] if it satisfies the following condition

$$\Phi'(u)[\Phi K]S - \Phi'(u)[\Phi S]K - \Phi\{\Phi'(u)[K]S - \Phi'(u)[S]K\} = 0 \tag{1.3}$$

for arbitrary vector functions $K, S \in \mathcal{A}^q$.

An equivalent definition of a hereditary symmetry operator $\Phi(u) : \mathcal{A}^q \to \mathcal{A}^q$ is that besides the linearity of $\Phi(u)$, its Nijenhuis torsion [15, 16] $N_\Phi(K, S)$ vanishes for all $K, S \in \mathcal{A}^q$, i.e.

$$\begin{aligned} N_\Phi(K, S) &:= [\Phi K, \Phi S] - \Phi[\Phi K, S] - \Phi[K, \Phi S] + \Phi^2[K, S] \\ &= (L_{\Phi S}\Phi)K - \Phi(L_S \Phi)K = 0 \end{aligned} \tag{1.4}$$

where a Lie derivative $L_K\Phi$ of $\Phi(u) : \mathcal{A}^q \to \mathcal{A}^q$ with respect to $K \in \mathcal{A}^q$ is given by

$$L_K\Phi = \Phi'[K] - [K', \Phi] \tag{1.5}$$

or more precisely,

$$(L_K\Phi)S = \Phi'(u)[K]S - K'(u)[\Phi S] + \Phi K'(u)[S] \qquad S \in \mathcal{A}^q. \tag{1.6}$$

If a hereditary symmetry operator $\Phi(u)$ has a zero Lie derivative $L_K\Phi = 0$ with respect to $K \in \mathcal{A}^q$, then we have (e.g., see [14, 17])

$$[\Phi^m K, \Phi^n K] = 0 \qquad m, n \geqslant 0. \tag{1.7}$$

Therefore each system of evolution equations among the hierarchy

$$u_t = \Phi^n K \qquad n \geqslant 0 \tag{1.8}$$

has infinitely many commuting symmetries $\Phi^m K$, $m \geqslant 0$. Such a vector field $K \in \mathcal{A}^q$ may often be chosen as u_x, which will be seen later on.

The next section will examine two models of candidates for hereditary symmetry operators. It will then go on to exhibit concrete examples of the general cases established in section 3. Finally, section 4 will provide us with a summary and some concluding remarks.

2. Extending hereditary symmetry operators

Let us assume that

$$\begin{aligned} &u_k = (u_k^1, \ldots, u_k^q)^{\mathrm{T}} \qquad 1 \leqslant k \leqslant N \\ &u = (u_1^{\mathrm{T}}, \ldots, u_N^{\mathrm{T}})^{\mathrm{T}} = (u_1^1, \ldots, u_1^q, \ldots, u_N^1, \ldots, u_N^q)^{\mathrm{T}}. \end{aligned}$$

Throughout this paper, we need the following condition

$$\Phi_k'(u_k) = \Phi_l'(u_l) \qquad 1 \leqslant k, l \leqslant N \tag{2.1}$$

for a set of operators $\Phi_k(u_k): \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k),\ 1 \leqslant k \leqslant N$. This reflects a kind of linearity property of the operators with respect to the dependent variables u_k, $1 \leqslant k \leqslant N$. We point out that such sets of operators $\Phi_k(u_k)$ do exist. Some examples will be given in the next section.

Let us consider the first form of candidates for hereditary symmetry operators

$$\Phi(u) = \left(\sum_{k=1}^{N} c_{ij}^{k} \Phi_k(u_k) \right)_{N\times N} \tag{2.2}$$

where $\{c_{ij}^k \mid i, j, k = 1, 2, \ldots, N\}$ is a set of given constants. Apparently we can define a linear operator

$$\Phi(u) : \underbrace{\mathcal{A}^q(u) \times \cdots \times \mathcal{A}^q(u)}_{N} \to \underbrace{\mathcal{A}^q(u) \times \cdots \times \mathcal{A}^q(u)}_{N}$$

where a vector function of $\mathcal{A}^q(u)$ depends on all the dependent variables $u_1, \ldots, u_N$, not just certain dependent variable u_k.

Theorem 2.1. (i) If all $\Phi_k(u_k) : \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k),\ 1 \leqslant k \leqslant N$, are hereditary symmetry operators satisfying the linearity condition (2.1) and the constants c_{ij}^k, $1 \leqslant i, j, k \leqslant N$, satisfy the following coupled condition

$$\sum_{k=1}^{N} c_{ij}^k c_{kn}^l = \sum_{k=1}^{N} c_{ik}^l c_{kj}^n = \sum_{k=1}^{N} c_{in}^k c_{kj}^l \qquad 1 \leqslant i, j, l, n \leqslant N \tag{2.3}$$

then the operator $\Phi(u) : \mathcal{A}^{Nq}(u) \to \mathcal{A}^{Nq}(u)$ defined by (2.2) is a hereditary symmetry operator.

(ii) If $L_{u_{kx}}\Phi_k = 0$ for all $\Phi_k(u_k)$, $1 \leqslant k \leqslant N$, then $L_{u_x}\Phi = 0$.

Proof. We only need to prove that $\Phi(u)$ satisfies the hereditary condition (1.3), because the proof of the rest of the requirements is obvious. Noting that $\mathcal{A}^q(u)$ is composed of column vector functions, we may assume for $K, S \in \mathcal{A}^{Nq}(u)$ that

$$K = (K_1^{\mathrm{T}}, \ldots, K_N^{\mathrm{T}})^{\mathrm{T}} \qquad S = (S_1^{\mathrm{T}}, \ldots, S_N^{\mathrm{T}})^{\mathrm{T}} \qquad K_i, S_i \in \mathcal{A}^q(u) \qquad 1 \leqslant i \leqslant N$$

and we often need to write $(X)_i = X_i$, $1 \leqslant i \leqslant N$, when a vector function $X \in \mathcal{A}^{Nq}(u)$ itself is complicated. In this way we have

$$\Phi K = ((\Phi K)_1^{\mathrm{T}}, \ldots, (\Phi K)_N^{\mathrm{T}})^{\mathrm{T}} \qquad (\Phi K)_i = \sum_{l,n=1}^{N} c_{in}^l \Phi_l(u_l) K_n \qquad 1 \leqslant i \leqslant N$$

$$\Phi'(u)[\Phi K] = \left(\sum_{k=1}^{N} c_{ij}^k \Phi_k'(u_k) \left[\sum_{l,n=1}^{N} c_{kn}^l \Phi_l(u_l) K_n \right] \right)_{N\times N}$$

$$(\Phi'(u)[\Phi K]S)_i = \sum_{j,k,l,n=1}^{N} c_{ij}^k c_{kn}^l \Phi_k'(u_k)[\Phi_l(u_l)K_n]S_j \qquad 1 \leqslant i \leqslant N$$

$$(\Phi\Phi'(u)[K]S)_i = \sum_{j,k,l,n=1}^{N} c_{ik}^l c_{kj}^n \Phi_l(u_l)\Phi_n'(u_n)[K_n]S_j \qquad 1 \leqslant i \leqslant N.$$

Therefore by the linearity condition (2.1), we can obtain

$$\begin{aligned}
&(\Phi'(u)[\Phi K]S - \Phi'(u)[\Phi S]K - \Phi\{\Phi'(u)[K]S - \Phi'(u)[S]K\})_i \\
&\quad = \sum_{j,l,n=1}^{N} f(i, j, l, n)\{\Phi_l'(u_l)[\Phi_l(u_l)K_n]S_j - \Phi_l'(u_l)[\Phi_l(u_l)S_j]K_n \\
&\qquad -\Phi_l(u_l)\{\Phi_l'(u_l)[K_n]S_j - \Phi_l'(u_l)[S_j]K_n\}\} \qquad 1 \leqslant i, j, n, l \leqslant N
\end{aligned} \tag{2.4}$$

where $f(i, j, l, n)$ is given by

$$f(i,j,l,n) := \sum_{k=1}^{N} c_{ij}^{k} c_{kn}^{l} = \sum_{k=1}^{N} c_{ik}^{l} c_{kj}^{n} = \sum_{k=1}^{N} c_{in}^{k} c_{kj}^{l} = \sum_{k=1}^{N} c_{ik}^{l} c_{kn}^{j} \qquad 1 \leqslant i, j, l, n \leqslant N.$$

This is well defined owing to (2.3). Actually the last equality above may be obtained by changing two indices n, j in the first equality of (2.3). Each term in the right-hand side of (2.4) is equal to zero because of the hereditary property of $\Phi_l(u_l)$, $1 \leqslant l \leqslant N$, and thus $\Phi(u)$ satisfies the hereditary condition (1.3). The proof is completed. □

Let us now consider the second form of candidates for hereditary symmetry operators

$$\Phi(u) = \begin{bmatrix} 0 & \cdots & 0 & \Phi_1(u_1) \\ E_q & \cdots & 0 & \Phi_2(u_2) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & E_q & \Phi_N(u_N) \end{bmatrix} \tag{2.5}$$

where the matrix E_q is the unit matrix of order q, i.e. $E_q = \mathrm{diag}\,\underbrace{(1, \ldots, 1)}_{q}$.

Theorem 2.2. (i) If the operators $\Phi_k(u_k) : \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k)$, $1 \leqslant k \leqslant N$, satisfy the linearity condition (2.1), then the operator $\Phi(u) : \mathcal{A}^{Nq}(u) \to \mathcal{A}^{Nq}(u)$ defined by (2.5) is hereditary if and only if the operators $\Phi_k(u_k)$, $1 \leqslant k \leqslant N$, are all hereditary.

(ii) The condition $L_{u_x}\Phi = 0$ holds if and only if all the conditions $L_{u_{kx}}\Phi_k = 0$, $1 \leqslant k \leqslant N$, hold.

Proof. Similarly noting that $\mathcal{A}^q(u)$ is composed of column vector functions, we may make the same assumption for $K, S \in \mathcal{A}^{Nq}(u)$

$$K = (K_1^{\mathrm{T}}, \ldots, K_N^{\mathrm{T}})^{\mathrm{T}} \qquad S = (S_1^{\mathrm{T}}, \ldots, S_N^{\mathrm{T}})^{\mathrm{T}} \qquad K_i, S_i \in \mathcal{A}^q(u) \qquad 1 \leqslant i \leqslant N.$$

Then we can obtain

$$\Phi'(u)[\Phi K]S = \begin{bmatrix} \Phi_1'(u_1)[\Phi_1 K_N]S_N \\ \Phi_2'(u_2)[K_1 + \Phi_2 K_N]S_N \\ \vdots \\ \Phi_N'(u_N)[K_{N-1} + \Phi_N K_N]S_N \end{bmatrix}$$

$$\Phi\Phi'(u)[K]S = \begin{bmatrix} \Phi_1\Phi_N'(u_N)[K_N]S_N \\ \Phi_1'(u_1)[K_1]S_N + \Phi_2\Phi_N'(u_N)[K_N]S_N \\ \vdots \\ \Phi_{N-1}'(u_{N-1})[K_{N-1}]S_N + \Phi_N\Phi_N'(u_N)[K_N]S_N \end{bmatrix}$$

$$L_{u_x}\Phi = \begin{bmatrix} 0 & \cdots & 0 & \Phi_1'(u_1)[u_{1x}] - (\partial\Phi_1 - \Phi_1\partial) \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \Phi_N'(u_N)[u_{Nx}] - (\partial\Phi_N - \Phi_N\partial) \end{bmatrix}.$$

Based upon the above three equalities and the linearity condition (2.1), we can easily obtain the required results. So the proof is finished. □

3. Concrete examples

Basic scalar hereditary symmetry operators satisfying the linearity condition (2.1) can be one of the following two sets

$$\Phi_i(u_i) = \alpha_i + \beta_i \partial^2 + \gamma(\partial u_i \partial^{-1} + u_i) \qquad 1 \leqslant i \leqslant N \tag{3.1}$$

$$\Phi_i(u_i) = \alpha_i \partial + \gamma(u_{ix} \partial^{-1} + u_i) \qquad 1 \leqslant i \leqslant N \tag{3.2}$$

where $\partial = \partial/\partial x$ and $\alpha_i, \beta_i, \gamma$ are arbitrary constants. Matrix hereditary symmetry operators satisfying the linearity condition (2.1) may be chosen and some of these examples have been given in [18–21]. Later on we will see two special examples while discussing extension problems. On the other hand, such sets of hereditary symmetry operators may be generated directly from the above operators by theorems 2.1 and 2.2 in the previous section or by perturbation around solutions as in [22, 23]. Note that all the above hereditary symmetry operators satisfy $L_{u_{ix}} \Phi_i = 0$, $1 \leqslant i \leqslant N$. Therefore among the corresponding hierarchy $u_t = \Phi^n u_x$, $n \geqslant 0$, each system of evolution equations has infinitely many commuting symmetries, because we have $[\Phi^m u_x, \Phi^n u_x] = 0$ if $\Phi(u)$ is hereditary.

3.1. Hereditary symmetry operators of the first form

Example 1. Let us choose

$$c_{ij}^k = f(i)g(j)g(k) \qquad 1 \leqslant i, j, k \leqslant N \tag{3.3}$$

where f, g may be arbitrary functions. The set of constants $\{c_{ij}^k\}$ satisfies the coupled condition (2.3) and thus the corresponding operator $\Phi(u)$ defined by (2.2) is hereditary if each $\Phi_k(u_k)$ is hereditary and the linearity condition (2.1) holds. In particular, upon choosing $f(1) = g(1) = 1$, $g(2) = 2$, $f(2) = -3$, we have the following special hereditary symmetry operator

$$\Phi(u) = \begin{bmatrix} \Phi_1(u_1) + 2\Phi_2(u_2) & 2\Phi_1(u_1) + 4\Phi_2(u_2) \\ -3\Phi_1(u_1) - 6\Phi_2(u_2) & -6\Phi_1(u_1) - 12\Phi_2(u_2) \end{bmatrix}$$

where we require that $\Phi_1(u_1)$ and $\Phi_2(u_2)$ are hereditary and that $\Phi_1'(u_1) = \Phi_2'(u_2)$. The second row of this operator is obtained by multiplying the first row by a constant -3 and so the operator is trivial. As the result of the same fact, all hereditary symmetry operators resulted from (3.3) are trivial.

Let us now choose

$$c_{ij}^k = \delta_{kl} \qquad l = i + j - p \pmod N \tag{3.4}$$

where $1 \leqslant p \leqslant N$ is fixed and δ_{kl} denotes the Kronecker symbol again. The corresponding operator defined by (2.2) becomes

$$\Phi(u) = \begin{bmatrix} \Phi_{2-p}(u_{2-p}) & \Phi_{1-p}(u_{1-p}) & \cdots & \Phi_{N-p+1}(u_{N-p+1}) \\ \Phi_{3-p}(u_{3-p}) & \Phi_{2-p}(u_{2-p}) & \cdots & \Phi_{N-p+2}(u_{N-p+2}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{N-p+1}(u_{N-p+1}) & \Phi_{N-p+2}(u_{N-p+2}) & \cdots & \Phi_{2N-p}(u_{2N-p}) \end{bmatrix} \tag{3.5}$$

where we need to use $\Phi_i(u_i) = \Phi_j(u_j)$ if $i = j \pmod N$ to determine the operators involved, for example, $\Phi_{2-p}(u_{2-p}) = \Phi_N(u_N)$ when $p = 2$.

It can be proved that the coupled condition (2.3) requires $N = 2$. Thus, among the above operators, we have only two candidates of hereditary symmetry operators satisfying (2.3)

$$\Phi(u) = \begin{bmatrix} \Phi_1(u_1) & \Phi_2(u_2) \\ \Phi_2(u_2) & \Phi_1(u_1) \end{bmatrix} \qquad \Phi(u) = \begin{bmatrix} \Phi_2(u_2) & \Phi_1(u_1) \\ \Phi_1(u_1) & \Phi_2(u_2) \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.6}$$

Note that here u_1 and u_2 may be vector functions. These two operators are symmetric and thus they can be diagonalizable. Actually they can be diagonalized by a linear transformation of the potentials u_1 and u_2. Therefore, they are also trivial. What we show above is that there is no interesting hereditary symmetry operator among the operators defined by (3.5).

Example 2. Let us choose

$$c_{ij}^k = \delta_{kl} \qquad l = i - j + p \pmod N \tag{3.7}$$

where $1 \leqslant p \leqslant N$ is also fixed and δ_{kl} still denotes the Kronecker symbol. In this case, we have

$$\sum_{k=1}^N c_{ij}^k c_{kn}^l = \sum_{k=1}^N c_{ik}^l c_{kj}^n = \sum_{k=1}^N c_{in}^k c_{kj}^l = \begin{cases} 1 & \text{when } i - j - n - l + 2p = 0 \pmod N \\ 0 & \text{otherwise} \end{cases}$$

which implies that the coupled condition (2.3) automatically holds. Thus we have a set of candidates for hereditary symmetry operators

$$\Phi(u) = \begin{bmatrix} \Phi_p(u_p) & \Phi_{p-1}(u_{p-1}) & \cdots & \Phi_1(u_1) & \Phi_N(u_N) & \cdots & \Phi_{p+1}(u_{p+1}) \\ \Phi_{p+1}(u_{p+1}) & \Phi_p(u_p) & \ddots & \ddots & \Phi_1(u_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \Phi_N(u_N) \\ \Phi_N(u_N) & & \ddots & \ddots & \ddots & \ddots & \Phi_1(u_1) \\ \Phi_1(u_1) & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \Phi_N(u_N) & & \ddots & \Phi_p(u_p) & \Phi_{p-1}(u_{p-1}) \\ \Phi_{p-1}(u_{p-1}) & \cdots & \Phi_1(u_1) & \Phi_N(u_N) & \cdots & \Phi_{p+1}(u_{p+1}) & \Phi_p(u_p) \end{bmatrix} \tag{3.8}$$

where we also need to use $\Phi_i(u_i) = \Phi_j(u_j)$ if $i = j \pmod N$ to determine the operators involved. In particular, we can obtain a candidate of hereditary symmetry operators

$$\Phi(u) = \begin{bmatrix} \Phi_1(u_1) & \Phi_N(u_N) & \cdots & \Phi_2(u_2) \\ \Phi_2(u_2) & \Phi_1(u_1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \Phi_N(u_N) \\ \Phi_N(u_N) & \Phi_{N-1}(u_{N-1}) & \cdots & \Phi_1(u_1) \end{bmatrix}. \tag{3.9}$$

The $N = 3$ case of the above operator with the scalar operators

$$\Phi_i(u_i) = \beta_i \partial^2 + (\partial u_i \partial^{-1} + u_i) \qquad 1 \leqslant i \leqslant 3$$

gives a hierarchy of nonlinear systems $u_t = (\Phi(u))^n u_x$, $n \geqslant 1$, among which the first nonlinear system reads as

$$\begin{cases} u_{1t} = \beta_1 u_{1xxx} + \beta_3 u_{2xxx} + \beta_2 u_{3xxx} + 3u_1 u_{1x} + 3(u_2 u_3)_x \\ u_{2t} = \beta_2 u_{1xxx} + \beta_1 u_{2xxx} + \beta_3 u_{3xxx} + 3u_3 u_{3x} + 3(u_1 u_2)_x \\ u_{3t} = \beta_3 u_{1xxx} + \beta_2 u_{2xxx} + \beta_1 u_{3xxx} + 3u_2 u_{2x} + 3(u_1 u_3)_x. \end{cases} \tag{3.10}$$

This system is not symmetric with respect to u_1, u_2, u_3, and generally it cannot be separated under a real linear transformation of the potentials u_1, u_2, u_3. One of the reasons is that the matrix

$$A = \begin{bmatrix} \beta_1 & \beta_3 & \beta_2 \\ \beta_2 & \beta_1 & \beta_3 \\ \beta_3 & \beta_2 & \beta_1 \end{bmatrix}$$

cannot always be diagonalized for all values of $\beta_1, \beta_2, \beta_3$. When $u_1 = u_2 = u_3$, the system is reduced to the KdV equation up to a constant coefficient. It also provides an example of the general systems discussed by Gürses and Karasu [24].

Example 3. We choose

$$c_{ij}^k = \delta_{i-j,k-p} \tag{3.11}$$

where p is an integer and δ_{kl} denotes the Kronecker symbol. For two cases of $2-N \leqslant p \leqslant 1$ and $N \leqslant p \leqslant 2N-1$, the coupled condition (2.3) can be satisfied, because we have

$$\sum_{k=1}^N c_{ij}^k c_{kn}^l = \sum_{k=1}^N c_{ik}^l c_{kj}^n = \sum_{k=1}^N c_{in}^k c_{kj}^l = \begin{cases} 1 & \text{when } i-j-n-l+2p=0 \\ 0 & \text{otherwise.} \end{cases}$$

We should note in proving the above equality that we have

$$1 \leqslant i-j+p = n+l-p \leqslant N \qquad 1 \leqslant i-l+p = n+j-p \leqslant N$$
$$1 \leqslant i-n+p = j+l-p \leqslant N$$

when $i-j-n-l+2p=0$. However, for the case of $1 < p < N$, upon choosing $i = n = N$, $j = p+1$, $l = p-1$, we have

$$\sum_{k=1}^N c_{ij}^k c_{kn}^l = 1 \qquad \sum_{k=1}^N c_{ik}^l c_{kj}^n = 0$$

and thus the coupled condition (2.3) cannot be satisfied.

Note that when $p < 2-N$ or $p > 2N-1$, the resulting operators are all zero operators. Therefore, we can obtain only two sets of candidates for hereditary symmetry operators

$$\Phi(u) = \begin{bmatrix} \Phi_p(u_p) & & & 0 \\ \Phi_{p+1}(u_{p+1}) & \ddots & & \\ \vdots & \ddots & \ddots & \\ \Phi_{p+N-1}(u_{p+N-1}) & \cdots & \Phi_{p+1}(u_{p+1}) & \Phi_p(u_p) \end{bmatrix} \qquad 2-N \leqslant p \leqslant 1 \tag{3.12}$$

$$\Phi(u) = \begin{bmatrix} \Phi_p(u_p) & \Phi_{p-1}(u_{p-1}) & \cdots & \Phi_{p-N+1}(u_{p-N+1}) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \Phi_{p-1}(u_{p-1}) \\ 0 & & & \Phi_p(u_p) \end{bmatrix} \qquad N \leqslant p \leqslant 2N-1 \tag{3.13}$$

where we accept that $\Phi_i(u_i) = 0$ if $i \leqslant 0$ or $i \geqslant N+1$. These two sets of operators can be linked by a transformation $(u_1, u_2, \ldots, u_N) \leftrightarrow (u_N, u_{N-1}, \ldots, u_1)$.

When we take

$$\Phi_i(u_i) = \alpha_i \partial^2 + 2(\partial u_i \partial^{-1} + u_i) \qquad 1 \leqslant i \leqslant N$$

where α_i, $1 \leqslant i \leqslant N$, are arbitrary constants, as basic hereditary symmetry operators, we obtain N hierarchies of nonlinear systems of KdV type starting from the operators in (3.12). A special choice with $\alpha_1 = 1$, $\alpha_i = 0$, $2 \leqslant i \leqslant N$, and $p = 1$ leads to the perturbation systems of the KdV equation generated from perturbation around solutions in [22]. Another special choice with $N = 2$ and $p = 1$ leads to the following system

$$\begin{cases} u_{1t} = \alpha_1 u_{1xxx} + 6u_1 u_{1x} \\ u_{2t} = \alpha_2 u_{1xxx} + \alpha_1 u_{2xxx} + 6(u_1 u_2)_x. \end{cases} \tag{3.14}$$

We can also choose a pair of hereditary symmetry operators in [25]

$$\Phi_1(u_1) = \begin{bmatrix} u^1_{1x}\partial^{-1} + 2u^1_1 & u^2_1 + \alpha\partial \\ u^2_{1x}\partial^{-1} + u^2_1 - \alpha\partial & 0 \end{bmatrix}$$
$$\Phi_2(u_2) = \begin{bmatrix} u^1_{2x}\partial^{-1} + 2u^1_2 & u^2_2 + \beta\partial \\ u^2_{2x}\partial^{-1} + u^2_2 - \beta\partial & 0 \end{bmatrix} \tag{3.15}$$

as basic hereditary symmetry operators with $u_1 = (u^1_1, u^2_1)^{\mathrm{T}}$ and $u_2 = (u^1_2, u^2_2)^{\mathrm{T}}$ and two arbitrary constants α and β. Then we can obtain a 4×4 matrix hereditary symmetry operator

$$\Phi(u) = \begin{bmatrix} u_{1x}\partial^{-1} + 2u_1 & u_2 + \alpha\partial & 0 & 0 \\ u_{2x}\partial^{-1} + u_2 - \alpha\partial & 0 & 0 & 0 \\ u_{3x}\partial^{-1} + 2u_3 & u_4 + \beta\partial & u_{1x}\partial^{-1} + 2u_1 & u_2 + \alpha\partial \\ u_{4x}\partial^{-1} + u_4 - \beta\partial & 0 & u_{2x}\partial^{-1} + u_2 - \alpha\partial & 0 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.16}$$

with two arbitrary constants α and β. Note that we rename the dependent variables $u^1_1, u^2_1, u^1_2, u^2_2$ as u_1, u_2, u_3, u_4, respectively. The first nonlinear system in the hierarchy $u_t = (\Phi(u))^n u_x$, $n \geqslant 1$, is the following

$$\begin{cases} u_{1t} = \alpha u_{2xx} + 3u_1 u_{1x} + u_2 u_{2x} \\ u_{2t} = -\alpha u_{1xx} + (u_1 u_2)_x \\ u_{3t} = \beta u_{2xx} + \alpha u_{4xx} + 3(u_1 u_3)_x + (u_2 u_4)_x \\ u_{4t} = -\beta u_{1xx} - \alpha u_{3xx} + (u_1 u_4)_x + (u_2 u_3)_x. \end{cases} \tag{3.17}$$

This is of different type from that discussed in [26] because of the terms of the second derivatives of potentials.

3.2. Hereditary symmetry operators of the second form

Example 4. Let $\Phi(u)$ be defined by (2.5). The first non-trivial candidate of integrable systems among the hierarchy $u_t = (\Phi(u))^n u_x$, $n \geqslant 0$, reads as

$$u_t = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_t = \begin{bmatrix} \Phi_1(u_1)u_{Nx} \\ u_{1x} + \Phi_2(u_2)u_{Nx} \\ \vdots \\ u_{N-1,x} + \Phi_N(u_N)u_{Nx} \end{bmatrix}. \tag{3.18}$$

If we choose the basic scalar hereditary symmetry operators as follows

$$\Phi_i(u_i) = -\tfrac{1}{4}\partial^2 + (\partial u_i \partial^{-1} + u_i) \qquad 1 \leqslant i \leqslant N$$

then the corresponding hereditary symmetry operator $\Phi(u)$ determined by (2.5) becomes

$$\Phi(u) = \begin{bmatrix} 0 & \cdots & 0 & -\frac{1}{4}\partial^2 + (\partial u_1 \partial^{-1} + u_1) \\ 1 & \cdots & 0 & -\frac{1}{4}\partial^2 + (\partial u_2 \partial^{-1} + u_2) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\frac{1}{4}\partial^2 + (\partial u_N \partial^{-1} + u_N) \end{bmatrix}. \tag{3.19}$$

This generates the coupled KdV systems [18, 27].

If we choose the basic scalar hereditary symmetry operators defined by (3.2), then the corresponding hereditary symmetry operator contains all hereditary symmetry operators appearing in [19–21]. A special example gives a hereditary symmetry operator

$$\Phi(u) = \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha_1\partial + u_{1x}\partial^{-1} + u_1 \\ 1 & 0 & 0 & 0 & \alpha_2\partial + u_{2x}\partial^{-1} + u_2 \\ 0 & 1 & 0 & 0 & \alpha_3\partial + u_{3x}\partial^{-1} + u_3 \\ 0 & 0 & 1 & 0 & \alpha_4\partial + u_{4x}\partial^{-1} + u_4 \\ 0 & 0 & 0 & 1 & \alpha_5\partial + u_{5x}\partial^{-1} + u_5 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \tag{3.20}$$

and a nonlinear system

$$\begin{cases} u_{1t} = \alpha_1 u_{5xx} + (u_1u_5)_x \\ u_{2t} = u_{1x} + \alpha_2 u_{5xx} + (u_2u_5)_x \\ u_{3t} = u_{2x} + \alpha_3 u_{5xx} + (u_3u_5)_x \\ u_{4t} = u_{3x} + \alpha_4 u_{5xx} + (u_4u_5)_x \\ u_{5t} = u_{4x} + \alpha_5 u_{5xx} + 2u_5u_{5x} \end{cases} \tag{3.21}$$

with five arbitrary constants $\alpha_i,\ 1 \leqslant i \leqslant 5$.

Example 5. Let us choose another pair of 2×2 matrix operators

$$\Phi_1(u_1) = \begin{bmatrix} 0 & \beta_1\partial + \gamma(u^1_{1x}\partial^{-1} + u^1_1) \\ \alpha_1 & \beta_2\partial + \gamma(u^2_{1x}\partial^{-1} + u^2_1) \end{bmatrix} \qquad \Phi_2(u_2) = \begin{bmatrix} 0 & \beta_3\partial + \gamma(u^1_{2x}\partial^{-1} + u^1_2) \\ \alpha_2 & \beta_4\partial + \gamma(u^2_{2x}\partial^{-1} + u^2_2) \end{bmatrix}$$

as basic hereditary symmetry operators with $u_1 = (u^1_1, u^2_1)^{\mathrm{T}}$ and $u_2 = (u^1_2, u^2_2)^{\mathrm{T}}$. Then by theorem 2.2, we obtain a 4×4 matrix hereditary symmetry operator

$$\Phi(u) = \begin{bmatrix} 0 & 0 & 0 & \beta_1\partial + \gamma(u_{1x}\partial^{-1} + u_1) \\ 0 & 0 & \alpha_1 & \beta_2\partial + \gamma(u_{2x}\partial^{-1} + u_2) \\ 1 & 0 & 0 & \beta_3\partial + \gamma(u_{3x}\partial^{-1} + u_3) \\ 0 & 1 & \alpha_2 & \beta_4\partial + \gamma(u_{4x}\partial^{-1} + u_4) \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.22}$$

where $\alpha_i, \beta_i, \gamma$ are arbitrary constants and we rename the dependent variables $u^1_1, u^2_1, u^1_2, u^2_2$ as u_1, u_2, u_3, u_4, respectively. The first nonlinear system from the corresponding hierarchy is the following

$$\begin{cases} u_{1t} = \beta_1 u_{4xx} + \gamma(u_1u_4)_x \\ u_{2t} = \alpha_1 u_{3x} + \beta_2 u_{4xx} + \gamma(u_2u_4)_x \\ u_{3t} = u_{1x} + \beta_3 u_{4xx} + \gamma(u_3u_4)_x \\ u_{4t} = u_{2x} + \alpha_2 u_{3x} + \beta_4 u_{4xx} + 2\gamma u_4u_{4x}. \end{cases} \tag{3.23}$$

This system is reduced to the Burgers equation up to a constant coefficient, if we make a special choice

$$u_1 = u_2 = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \qquad u_3 = u_4 \qquad \beta_3 = \beta_4.$$

Let us next choose the following three 2×2 matrix operators in [25]

$$\Phi_i(u_i) = \begin{bmatrix} u^2_i + \alpha_i\partial & u^1_{ix}\partial^{-1} + 2u^1_i \\ 0 & u^2_{ix}\partial^{-1} + u^2_i - \alpha_i\partial \end{bmatrix} \qquad 1 \leqslant i \leqslant 3 \tag{3.24}$$

as basic hereditary symmetry operators with $u_i = (u^1_i, u^2_i)^{\mathrm{T}}$, $1 \leqslant i \leqslant 3$. It is quite interesting to observe that the above hereditary symmetry operators can be obtained by interchanging two columns of the hereditary symmetry operators in (3.15). Through theorem 2.2, we obtain a 6×6 matrix hereditary symmetry operator

$$\Phi(u) = \begin{bmatrix} 0 & 0 & 0 & 0 & u_2+\alpha_1\partial & u_{1x}\partial^{-1}+2u_1 \\ 0 & 0 & 0 & 0 & 0 & u_{2x}\partial^{-1}+u_2-\alpha_1\partial \\ 1 & 0 & 0 & 0 & u_4+\alpha_2\partial & u_{3x}\partial^{-1}+2u_3 \\ 0 & 1 & 0 & 0 & 0 & u_{4x}\partial^{-1}+u_4-\alpha_2\partial \\ 0 & 0 & 1 & 0 & u_6+\alpha_3\partial & u_{5x}\partial^{-1}+2u_5 \\ 0 & 0 & 0 & 1 & 0 & u_{6x}\partial^{-1}+u_6-\alpha_3\partial \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \tag{3.25}$$

where α_i, $1 \leqslant i \leqslant 3$, are arbitrary constants and we rename the dependent variables $u_1^1, u_1^2, u_2^1, u_2^2, u_3^1, u_3^2$ as $u_1, u_2, u_3, u_4, u_5, u_6$, respectively. The first nonlinear system of the corresponding hierarchy reads as

$$\begin{cases} u_{1t} = \alpha_1 u_{5xx} + u_2 u_{5x} + (u_1u_6)_x + u_1 u_{6x} \\ u_{2t} = -\alpha_1 u_{6xx} + (u_2u_6)_x \\ u_{3t} = u_{1x} + \alpha_2 u_{5xx} + u_4 u_{5x} + (u_3u_6)_x + u_3 u_{6x} \\ u_{4t} = u_{2x} - \alpha_2 u_{6xx} + (u_4u_6)_x \\ u_{5t} = u_{3x} + \alpha_3 u_{5xx} + 2(u_5u_6)_x \\ u_{6t} = u_{4x} - \alpha_3 u_{6xx} + 2u_6u_{6x}. \end{cases} \tag{3.26}$$

This is another different example from the systems discussed by Svinolupov in [26].

4. Conclusions and remarks

Two models of candidates for hereditary symmetry operators are analysed and some possible basic hereditary symmetry operators are also given. Therefore, according to the conditions in theorems 2.1 and 2.2, many concrete nonlinear systems of evolution equations possessing infinitely many symmetries may be generated from various hereditary symmetry operators having a zero Lie derivative with respect to u_x. Some particular cases are carefully discussed, along with several corresponding nonlinear systems.

Our results provide a direct way to extend hereditary symmetry operators. New resulting hereditary symmetry operators, for example the hereditary symmetry operators shown in (3.16), (3.22) and (3.25), can also be chosen as basic ones satisfying the linearity condition (2.1), and then more complicated hereditary symmetry operators can be generated by our idea of construction. Note that x may be a vector and no condition has been imposed on the spatial dimension while examining two forms of candidates for hereditary symmetry operators. Therefore, the idea is also valid for the case of high spatial dimensions, which will be reported elsewhere. On the other hand, we hope that there will appear more concrete examples satisfying (2.3) and more concrete models of hereditary symmetry operators.

It is worth pointing out that the coupled condition (2.3) is only sufficient but not necessary. We may have counterexamples. For example, a counterexample can be the following

$$\Phi(u) = \begin{bmatrix} \Phi_1(u_1) & 0 \\ \Phi_2(u_2) + a\Phi_1(u_1) & \Phi_1(u_1) \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.1}$$

$$\Phi_i(u_i) = \delta_{i1}\partial^2 + 2(\partial u_i \partial^{-1} + u_i) \qquad i = 1, 2 \tag{4.2}$$

with an arbitrary non-zero constant a. In fact, for this operator $\Phi(u)$ we have

$$(c_{ij}^1) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \qquad (c_{ij}^2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

If we choose $i = l = n = 1,\ j = 2$, then we obtain

$$\sum_{k=1}^{2} c_{ij}^k c_{kn}^l = 0 \qquad \sum_{k=1}^{2} c_{ik}^l c_{kj}^n = a$$

and thus the first equality in the coupled condition (2.3) is not satisfied. But the operator $\Phi(u)$ defined by (4.1) and (4.2) is hereditary, which may be directly proved.

The coupled condition (2.3) may also be viewed as a condition on a finite-dimensional algebra with a basis $e_1, e_2, \ldots, e_N$ and an operation

$$e_i * e_j = \sum_{k=1}^{N} c_{ij}^k e_k \qquad 1 \leqslant i, j \leqslant N. \tag{4.3}$$

However, we do not yet know much about such algebras.

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